Introduction to Jones Calculus

Abstract: Jones Calculus is a mathematical technique for describing the polarization state of light and calculating the evolution of the polarization state as light passes through optical devices. Any state of polarization can be described by a two-element *Jones vector*, and the linear operation of any optical device can be fully described by a 2×2 *Jones matrix*. A system of multiple devices can be straightforwardly modeled by multiplying the component Jones matrices to yield a single system Jones matrix.

A broadly useful representation for polarized light was invented in 1941 by R. Clark Jones. Like the method of Stokes parameters and Mueller matrices, the Jones method provides a mathematical description of the polarization state of light, as well as a means to calculate the effect that an optical device will have on input light of a given polarization state. The method of Jones is unique in that it deals with the instantaneous electric field, whereas the Stokes parameters describe a time-averaged optical signal. For this reason the Stokes/Mueller method is often chosen for use with light of rapidly and randomly changing polarization state, such as natural sunlight, while the Jones method is preferred when using coherent sources such as lasers.

Since light is composed of oscillating electric and magnetic fields, Jones reasoned that the most natural way to represent light is in terms of the electric field vector. When written as a column vector, this vector is known as a *Jones vector* and has the form

$$\vec{E} = \begin{bmatrix} E_x(t) \\ E_y(t) \end{bmatrix}, \qquad \text{(EQ 1)}$$

where $E_x(t)$ and $E_y(t)$ are the instantaneous scalar components of the electric field. Note that these values can be complex numbers, so both amplitude and phase information is present. Oftentimes, however, it is not necessary to know the exact amplitudes and phases of the vector components. Therefore Jones vectors can be normalized and common phase factors can be neglected. This results in a loss of information, but can greatly simplify expressions. For example, the following vectors contain varying degrees of information, but are all Jones vector representations for the same polarization state:

$$\begin{bmatrix} E_0 e^{i\phi} \\ E_0 e^{i\psi} \end{bmatrix} \rightarrow \begin{bmatrix} e^{i\phi} \\ e^{i\psi} \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ e^{i(\psi - \phi)} \end{bmatrix}. \qquad \text{(EQ 2)}$$

Note that a complex vector is said to be normalized when the dot product of the vector with its complex conjugate yields a value of unity.

It is most often the case that the basis for the Jones vector is chosen to be the horizontal and vertical linear polarization states. In this case the representations for these two states are

$$\vec{E}_h = \begin{bmatrix} E_x(t) \\ 0 \end{bmatrix} \text{ and } \vec{E}_v = \begin{bmatrix} 0 \\ E_y(t) \end{bmatrix}, \qquad \text{(EQ 3)}$$

or, in normalized form,

$$\vec{E}_h = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \vec{E}_v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \text{(EQ 4)}$$

where $\vec{E}_h$ and $\vec{E}_v$ represent horizontally and vertically polarized light, respectively. The sum of two coherent light beams is given by the sum of their corresponding Jones vector components, so the sum of $\overrightarrow{E_h}$ and $\overrightarrow{E_v}$ when $E_y(t) = E_x(t)$ is given by

$$\vec{E}_{45^\circ} = \begin{bmatrix} E_x(t) \\ E_x(t) \end{bmatrix} \Rightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \text{(EQ 5)}$$

where the arrow indicates normalization. Note that this is the representation of the polarization state in which the electric field is oriented at a 45 degree angle with respect to the basis states.

Two other common polarization states are right-circular and left-circular. In both cases the two components have equal amplitude, but for right circular the phase of the y-component leads the x-component by $\pi/2$, while for left circular it is the x-component that leads. Thus the Jones vector representation for right-circular is

$$\vec{E}_R = \begin{bmatrix} E_0 e^{i\phi} \\ E_0 e^{i(\phi - \pi/2)} \end{bmatrix}. \qquad \text{(EQ 6)}$$

Normalizing this expression and factoring out a constant phase factor of $e^{i\phi}$ yields

$$\vec{E}_R = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ e^{i\pi/2} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix}. \qquad \text{(EQ 7)}$$

Similarly, the normalized representation for left-circular light is

$$\vec{E}_L = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix}. \qquad \text{(EQ 8)}$$

Next consider a beam of light represented by the Jones vector

$$\vec{E}_i = \begin{bmatrix} E_{ix} \\ E_{iy} \end{bmatrix} \tag{EQ 9}$$

incident on an optical device. The light will interact with the device, and the new polarization state of the light upon exiting the device will be

$$\vec{E}_t = \begin{bmatrix} E_{tx} \\ E_{ty} \end{bmatrix}. \tag{EQ 10}$$

The coupling between these two vectors can be fully described by a set of four coefficients according the following pair of linear equations:

$$\begin{aligned} E_{tx} &= aE_{ix} + bE_{iy} \\ E_{ty} &= cE_{ix} + dE_{iy}. \end{aligned} \tag{EQ 11}$$

These two equations can be rewritten using matrix notation as

$$\vec{E}_t = \bar{J}\vec{E}_i \tag{EQ 12}$$

where

$$\bar{J} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{EQ 13}$$

is the *Jones matrix* of the optical device. A list of Jones matrices for some common optical devices appears in Table 1.

It is possible to represent the passage of a beam of light through multiple devices as the multiplication of Jones matrices. Note that the matrices do not commute, as illustrated by the following example. Let's assume a vertically polarized input signal, and look at its propagation through two devices, a linear polarizer oriented at $45°$ and a quarter-wave plate with its fast axis vertical. If the light passes through the polarizer first, followed by the wave plate, we have

$$\begin{aligned} \vec{E}_t &= \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ -i \end{bmatrix}, \end{aligned} \tag{EQ 14}$$

where we have neglected common amplitude and phase factors for simplicity. The output is right-circularly polarized. Now if the light passes through wave plate before the polarizer, the result is

$$\vec{E}_t = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{EQ 15}$$

which is light linearly polarized at $45°$.

While matrix multiplication is not commutative, it is associative, so a string of multiple Jones matrices representing several devices may be multiplied together yielding a single Jones matrix which describes the optical system as a whole. Therefore it is possible to condense the properties of N optical devices acting in series down to a single 2×2 matrix simply by multiplying the Jones matrices of the devices.

TABLE 1. Jones Matrices of Common Optical Devices

Vertical Linear Polarizer	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$	Right Circular Polarizer	$\frac{1}{2}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$
Horizontal Linear Polarizer	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	Left Circular Polarizer	$\frac{1}{2}\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}$
Linear Polarizer at 45°	$\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	Quarter-wave plate, fast axis vertical	$e^{i\pi/4}\begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}$
Lossless fiber transmission	$\begin{bmatrix} e^{i\phi}\cos\theta & -e^{-i\psi}\sin\theta \\ e^{i\psi}\sin\theta & e^{-i\phi}\cos\theta \end{bmatrix}$	Quarter-wave plate, fast axis horizontal	$e^{i\pi/4}\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$